Exercise 0 is probably well-known. Exercises 1 and 2 are important and elementary.

Exercice 0. Homological algebra reminder.
Recall that the shifted chain complex $C[1]_\bullet$ is defined by $C[1]_n = C_{n-1}$, and the mapping cone of a chain map $f\colon C_\bullet \to D_\bullet$ is a chain complex $C(f)_\bullet$ with $C(f)_n = D_n \oplus C_{n-1}$, its differential is given by $d(y, x) = (dy + fx, -dx)$.

1. Verify that $C(f)_\bullet$ is a chain complex fitting in a short exact sequence of chain complexes $D_\bullet \to C(f)_\bullet \to C[1]_\bullet$.
2. Show that f is a quasi-isomorphism if and only if $C(f)_\bullet$ is acyclic.
3. Show that f is a homotopy equivalence if and only if $C(f)_\bullet$ is contractible.
4. Find an example of an acyclic complex which is not contractible.

Exercice 1. Cyclic groups. Let C_n be a cyclic group of order n.

1. Prove that $\mathbb{Z}C_n$ is isomorphic to $\mathbb{Z}[t]/(t^n - 1)$.
2. Construct a free $\mathbb{Z}C_n$-resolution of $\mathbb{Z}$.
3. Compute $H_k(C_n; \mathbb{Z})$ for all $k \geq 0$.

Exercice 2. Universal covers. In the following cases describe a contractible CW-complex equipped with a free G-action on its cells, identify its cellular chain complex, and compute the homology groups $H_k(G; \mathbb{Z})$ for all $k \geq 0$.

1. $G = 1$, the trivial group.
2. $G = \mathbb{Z} \times \mathbb{Z}$.
3. $G = F(a, b)$, free group on two generators.

$\Diamond$ **Exercice 3. The bar resolution.** Let G be a group and $F_\bullet \to \mathbb{Z}$ be the augmented bar resolution where F_n is the free $\mathbb{Z}$-module on G^{n+1}. We write

$$\partial_i(g_0, g_1, \ldots, g_n) = (g_0, \ldots, \hat{g}_i, \ldots, g_n)$$

for the n-uple obtained by forgetting the i-th entry g_i, and construct then $d\colon F_n \to F_{n-1}$ by setting $d = \sum (-1)^i \partial_i$.

1. Show that the augmented bar resolution is a chain complex.
2. Show that it is acyclic by constructing a contracting homotopy of $\mathbb{Z}$-modules (see Brown's book, page 18 for more hints).
3. Let G act by left multiplication on G^{n+1} componentwise and show that $F_\bullet$ is a resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules. We choose $(1, g_1, g_1g_2, \ldots, g_1 \ldots g_n)$ as representative for each free orbit of generators and call $[g_1| \ldots |g_n]$ the class in the coinvariants.
4. Conclude that $H_*(G; \mathbb{Z})$ is the homology of the coinvariants of $F_\bullet$.

◊ **Exercice 4. The bar resolution, computations.** Let G be a group and $F_\bullet \to \mathbb{Z}$ be the augmented bar resolution.

1. Identify F_2, F_1 and F_0, as well as the differentials d_1 and d_2.
2. Identify the coinvariants $(F_2)_G$, $(F_1)_G$, and $(F_0)_G$, as well as the induced differentials in terms of the bar notation defined in the previous exercise.
3. Compute $H_0(G;\mathbb{Z})$ and $H_1(G;\mathbb{Z})$. (*Hint.* Define a map $(F_1)_G \to G_{ab}$ by sending $[g]$ to the class $g[G,G]$.)

◊ **Exercice 5. Hochschild complex.** Let k be a commutative ring, A a (unital) k-algebra with multiplication μ, and M an A-bimodule (the left and right actions commute : $a(mb) = (am)b$). We write $\otimes$ for the tensor product of k-modules.
Define $d_i \colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes(n-1)}$ for $0, 1 \leq i \leq n-1$, and n by the following formulas :

$$d_0(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n$$
$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$d_n(m \otimes a_1 \otimes \cdots \otimes a_n) = a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

The *Hochschild boundary* is the k-linear map $b = \sum(-1)^i d_i$.

1. Prove that $b \circ b = 0$. *Hint.* Prove that $d_i d_j = d_{j-1} d_i$ when $i < j$.
2. Write $C_\bullet(A;M)$ for the *Hochschild complex* and $C_\bullet(A) = C_\bullet(A;A)$. Show that this construction is functorial in M and A (define the bimodule structure you need here).
3. We define $HH_*(A) = H_*(C_\bullet(A))$. Compute $HH_0(A;M)$, $HH_0(A)$, and $HH_n(k)$ for all $n \geq 0$.

◊ **Exercice 6. Morita invariance for HH_0.** Denote by $M_r(R)$ the associative ring of $r \times r$-matrices with entries in the ring R. With the same notation as in Exercise 5 we want to prove that the trace map $M_r(R) \to R$ induces an isomorphism on HH_0 for any ring R. We call Tr the composition of the trace map, followed by abelianization $R \to R/[R,R]$.

1. Explain which ring k and k-algebra A of exercise 5 we use here.
2. Show that $[M_r(R), M_r(R)]$ is contained in the kernel of Tr.
3. Show that this kernel is generated by the elementary matrices $E_{ij}(r)$ with $i \neq j$ and diagonal matrices whose trace is in $[R,R]$.
4. Show that these generators all belong to $[M_r(R), M_r(R)]$.
5. Conclude that the trace map induces an isomorphism $HH_0(M_r(R)) \cong HH_0(R)$.

◊ indicates the weekly assignments you can (and should) volunteer for, in pairs.